Total screening and finite range forces from ultra-massive scalar fields

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Abstract

Force between static point particles coupled to a classical ultra-massive scalar field is calculated. The field potential is proportional to the modulus of the field. It turns out that the force exactly vanishes when the distance between the particles exceeds certain finite value. Moreover, each isolated particle is surrounded by a compact cloud of the scalar field that completely screens its scalar charge.

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1 Introduction

The paradigmatic classical forces between two elementary static charges in three-dimensional space are the Coulomb and Yukawa force, depending on whether the pertinent mediating field is massless or massive. In the case of free fields, or weak nonlinear fields for which field equations can be linearized, there is no other possibility. Nonlinearity of the field equation can result in a very different force, e.g., the constant force confining quark and anti-quark. Below we present another unusual force – it exactly vanishes when the distance between the charges exceeds certain finite value dependent on their strengths. The mediating field is a classical scalar field with a V-shaped potential, namely the signum-Gordon field. The name reflects the fact that the non-linear term in the field equation is given by the sign φ function, see Eq. (15) below.

Scalar fields have found many applications in physics. They appear in condensed matter physics as effective fields (scalar order parameters) in, e.g., Ginzburg – Landau type models of superconductors or atomic Bose – Einstein condensates. Effective scalar fields appear also in particle physics, e.g., in the Skyrme model of baryons. The discovery of the Higgs particle means that scalar fields are indispensable in fundamental theories of matter. There is a chance that scalar fields play important role in cosmology. Several scalar fields have been considered in connection with such phenomena as the inflation or the presence of dark matter , e.g., axions, chameleons, and K-fields [1]. For instance, the recent papers [2] are devoted to the so called classical fifth force due to various cosmological scalar fields.

Furthermore, interesting scalar fields appear also in a less spectacular, but physically very sound subject of continuous descriptions of discrete, nonlinear systems in the long wave limit. Here the classic example is the sinus-Gordon field associated with a long chain of harmonically coupled pendulums [3]. The signum-Gordon field considered in the present paper gives an effective, long-distance description of a system of coupled pendulums impacting on a stiff rod, and also of a system of coupled balls elastically bouncing from a floor in the constant gravitational field [4, 5]. Earlier it was considered as a model of pinning [6], and as a solvable model of decay of false vacuum [7]. The signum-Gordon field is the relativistic scalar field φ with the V-shaped field potential given by the absolute value of the field, $U(\varphi) = g|\varphi|$, where $g > 0$ is the self-coupling constant. Rather surprisingly, models with V-shaped potentials have turned out to be quite interesting. In particular, they support non-radiating compact oscillons [8], as well as compact Q-balls [9]. The compact Q-balls have been generalized to boson stars [10] by in-

cluding the gravity. Yet another model with a V-shaped potential was considered in connection with baby-skyrmions [11]. Generally speaking, we think that the signum-Gordon field represents a new class of scalar fields, which one may call the ultra-massive ones. The point is that in the three-dimensional space the massless or massive fields are characterized by, respectively, power-like or exponential approach to the vacuum value of the field, while in the signum-Gordon case the vacuum value is reached exactly, on a finite distance [4]. Such behavior is typical for all scalar fields with a V-shaped field potential.

The force between two point charges in the case of ultra-massive mediating field has not been investigated yet. Our main finding is that the force has a strictly finite range. Moreover, it turns out that a single static charge is surrounded by a compact spherical cloud of the scalar field which exactly vanishes outside certain ball centered at the charge – the presence of the charge can be felt only at close enough distances. It turns out that these effects are present also in the one-dimensional case. This is surprising because in the massive and massless cases the change of dimensionality of space significantly influences the asymptotic behavior of the field.

The plan of our paper is as follows. In Section 2 we recall the method of computing the forces based on the energy-momentum tensor. Section 3 is devoted to the proper field of a static point charge and to the force between two static charges in the three-dimensional space. In Section 4 we consider the one-dimensional case, where we can obtain exact analytic formula for the force. Section 5 contains a summary and remarks.

2 Preliminaries

The model we consider consists of the dynamical real scalar field φ interacting with two point-like classical particles at rest. The Lagrangian has the form

$$L = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - U(\varphi) + j\varphi, \tag{1}$$

where

$$j = \sum_{l=1}^{2} q_l \, \delta(\vec{x} - \vec{a}_l). \tag{2}$$

Here q_l is the scalar charge of the l-th particle and $\vec{a}_l$ its position in the space. The field potential has the form $U = g|\varphi|$ in the case of signum-Gordon model, and

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$U = m^2 \varphi^2/2$ in the case of free Klein – Gordon field. We use the $c = \hbar = 1$ units. Because the particles are kept at rest we omit kinetic terms for them. The Euler – Lagrange equation corresponding to (1) reads

$$\partial_\mu \partial^\mu \varphi + U'(\varphi) = j. \tag{3}$$

In order to find the force acting on the static charge q_1 we use the well-known method [12] based on computation of the total flux of momentum through a closed surface enclosing this charge with the other charge left outside. In particular, it can be the sphere centered at $\vec{a}_1$ with arbitrarily small radius $\epsilon > 0$. This flux gives the rate of transfer of momentum to the charge. It is equal to the force, in accordance with the Newton's formula $\vec{F} = d\vec{p}/dt$. The momentum density and its flux are given by the energy-momentum tensor for our system,

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \eta_{\mu\nu} L, \tag{4}$$

where $(\eta_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski space-time metric.

The total momentum $\vec{P}$ of the field is given by

$$P^i = -\int d^3x \, T_{0i} = -\int d^3x \, \partial_0 \varphi \partial_i \varphi. \tag{5}$$

We will consider only static fields, hence $P^i = 0$. The presence of the static charges breaks the translational symmetry. Therefore, instead of the continuity equation we have

$$\partial^\mu T_{\mu\nu} = -\varphi \partial_\nu j. \tag{6}$$

The total flux of the i-th component of the momentum across a surface Σ surrounding the charge q_1 is given by $\int_\Sigma d\Sigma^k T_{ki}$. Apart from the location of the charges the flux is conserved, $\partial_k T_{ki} = 0$, hence the total flux does not depend on the choice of Σ provided that in the space between two such surfaces there are no charges. The force exerted on the charge q_1 is equal to the total momentum flowing through the surface Σ enclosing this charge,

$$F^i = -\int_\Sigma d\Sigma^k \, T_{ki}, \tag{7}$$

where the surface element $d\vec{\Sigma}$ is directed outside the volume V enclosed by Σ. The incoming momentum is absorbed entirely by the charge because the momentum density of the field remains constant – in the case of static fields $T_{0i} = 0$. The

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main problem with formula (7) is that in order to make use of it we first have to find solution of the field equation (3). This can be a nontrivial task if the equation is nonlinear.

Equations (6), (7) give in the static case

$$F^i = -\int_V d^3x \; \varphi \partial_i j = \int_V d^3x \; \partial_i \varphi j, \tag{8}$$

where we have used the fact that the charge density j vanishes on Σ. Note that these formulas for the force are correct when the integrands are well-defined, and this is not the case for point charges.

In order to calculate the force we will use formula (7) with a conveniently chosen surface Σ. The term $j\varphi$ present in the Lagrangian L and entering T_{ik}, see formula (4), will vanish on Σ. In this way we avoid problems with the Coulomb type singularity of φ at the location of the point charge.

As an illustration and a check, let us compute the force between the two static point charges in the case of free scalar field with the potential $U(\varphi) = m^2\varphi^2/2$. The charge q_1 is located at the origin, while q_2 at the point $\vec{a} = (0, 0, a)$. Here $a > 0$ gives the distance between the charges. Equation (3) reads

$$\Delta\varphi - m^2\varphi = -q_1\delta(\vec{x}) - q_2\delta(\vec{x} - \vec{a}).$$

Its vanishing at the spatial infinity solution has the form

$$\varphi = q_1 \frac{e^{-mr}}{4\pi r} + q_2 \frac{e^{-m|\vec{x}-\vec{a}|}}{4\pi|\vec{x}-\vec{a}|}, \tag{9}$$

where $r = |\vec{x}|$. In order to compute the force exerted on the charge q_1 we use formula (7) with Σ the sphere of the small radius ϵ, $\epsilon \to 0$, centered at the charge q_1. The surface integral in (7) is simple in the spherical coordinates, especially in the limit $\epsilon \to 0$. It gives the following formula for the force:

$$F^1 = F^2 = 0, \quad F^3 = \frac{q_1 q_2}{4\pi}\left(\frac{m}{a} + \frac{1}{a^2}\right)e^{-ma}. \tag{10}$$

We see that the dependence on the distance a has the Yukawa form if $m^2 > 0$, and the Coulomb form if $m^2 = 0$. The charges of the same sign attract each other, and repel if they have opposite signs. The small distance behavior of the force is obtained by expanding with respect to the dimensionless product am:

$$F^3 = \frac{q_1 q_2}{4\pi a^2}\left[1 - \frac{(am)^2}{2} + \frac{(am)^3}{3} - \frac{(am)^4}{8} + \dots\right]. \tag{11}$$

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In the case of static electric charges coupled to the electromagnetic field the force can be calculated in the same manner. In particular, the electrostatic potential ψ also has the form (9), with $m = 0$ of course. However, there is the sign difference between the symmetric stress tensors. In the electrodynamics, the pertinent formula has the form [13]

$$T_{ik}^{el} = -E^i E^k - B^i B^k + \frac{1}{2}\delta_{ik}(\vec{E}^2 + \vec{B}^2).$$

In the electrostatic case $E^k = -\partial_k\psi$, $\vec{B} = 0$, and

$$T_{ik}^{el} = -\partial_k\psi\partial_k\psi + \frac{1}{2}\delta_{ik}\,\partial_l\psi\partial_l\psi,$$

while according to formula (4) for the free massless scalar field

$$T_{ik} = \partial_i\varphi\partial_k\varphi - \frac{1}{2}\delta_{ik}\,\partial_l\varphi\,\partial_l\varphi$$

(in the region of space in which $j = 0$). Therefore, in the case of electric charges we obtain the expected repulsion of charges of same sign.

Let us digress on the case the mediating scalar field is tachyonic, i.e., $m^2 < 0$. In the presence of the static charges q_1, q_2 the field has the form

$$\varphi = q_1\frac{\cos(|m|r)}{4\pi r} + q_2\frac{\cos(|m||\vec{x} - \vec{a}|)}{4\pi|\vec{x} - \vec{a}|}, \tag{12}$$

which can be obtained from formula (9) by inserting $m = i|m|$ and taking the real part. The force turns out to be equal to the real part of formula (10), i.e.,

$$F^1 = F^2 = 0, \quad F^3 = \frac{q_1 q_2}{4\pi}\left[\frac{|m|\sin(|m|a)}{a} + \frac{\cos(|m|a)}{a^2}\right]. \tag{13}$$

In this case the force decreases very slowly with the distance a, and also its sign is not constant. One should however remember that the field configuration (12) is unstable – even small perturbations can grow exponentially with time when m is imaginary, hence the significance of formula (13) is limited. The unstable modes have the length of the wave vector $\vec{k}$ bounded by $|m|$, $|\vec{k}| < |m|$. The corresponding wave length $\lambda = 2\pi/|\vec{k}|$ is bounded from below, $\lambda > \lambda_0 = 1/2\pi|m|$. If the distance a is much smaller than λ_0 the unstable modes may be regarded as almost homogeneous background field generating a force which is very weak in comparison with (13). Thus, formula (13) may be regarded as relevant when $a|m| \ll 1/2\pi$. Its expansion with respect to $a|m|$ has the form

$$F^3 = \frac{q_1 q_2}{4\pi a^2}\left[1 + \frac{(a|m|)^2}{2} - \frac{(a|m|)^4}{8} + \dots\right]. \tag{14}$$

3 The force in the signum-Gordon model

Exact calculation of the force mediated by a self-interacting scalar field most often is not possible because the pertinent exact static solutions of Eq. (3) are not available. In the next section we show that in the one-dimensional signum-Gordon model such solutions can be constructed. In the three-dimensional case the exact formula for the force is not known. Nevertheless, the very fact that the force exactly vanishes when the distance between the charges is large enough can be seen rather easily – it turns out that the proper fields of the particles have strictly finite range.

In the case of a single point charge $q > 0$ located at the origin the field equation has the form

$$\Delta \varphi = g \operatorname{sign} \varphi - q\, \delta(\vec{x}), \tag{15}$$

where the sign function has the values $+1$ or -1 for $\varphi \neq 0$, and $\operatorname{sign} 0 = 0$. Let us use an analogy between this equation and the Poisson equation of electrostatics. The δ term generates the Coulomb part of the field, $\varphi_c = q/4\pi r$, while the $g \operatorname{sign} \varphi$ term is interpreted as to an 'electric' charge density which is constant in the whole region in which φ has a constant sign. Close to the charge the Coulomb part dominates, hence the $\varphi > 0$. Here the nonlinear term effectively acts as a constant negative charge density $-g$. The vanishing 'electric' field $-\nabla \varphi$ is obtained when that charge exactly compensates the charge q. Thus, assuming the spherical symmetry, we have the condition

$$q = \frac{4\pi}{3} g R^3(q), \tag{16}$$

which determines the radius $R(q)$ of the ball in which we have a nontrivial field ϕ. Outside that ball the 'electrostatic' potential is put to zero by adding the appropriate constant, thus $\varphi(r) = 0$ for $r \geq R(q)$.

Of course, such a heuristic reasoning does not eliminate the necessity of finding the relevant solution φ of Eq. (15) inside that ball, and checking whether it can be matched with the vacuum field $\varphi = 0$. Assuming the spherical symmetry, φ obeys the equation

$$\partial_r^2 \varphi + \frac{2}{r} \partial_r \varphi = g \operatorname{sign} \varphi - q\, \delta(\vec{x}). \tag{17}$$

The positive solution that corresponds to the heuristic picture above has rather elementary, easy to guess form

$$\varphi_c(r) = \frac{g}{6} r^2 + \frac{q}{4\pi r} + c_0(q), \tag{18}$$

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where $c_0(q)$ is constant with respect to r, the subscript c stands for 'cloud'. This solution can be glued with the vacuum field $\varphi = 0$. The matching conditions, imposed at a certain radius R, have the standard form $\varphi_c(R) = 0$, $\partial_r \varphi_c(R) = 0$. They give

$$R = R(q) = \left(\frac{3q}{4\pi g} \right)^{1/3}, \quad c_0(q) = -\frac{3q}{8\pi R(q)}.$$

To summarize, the proper field of the located at the origin point charge q has the form

$$\varphi_q(r) = \left\{ \begin{array}{cc} \varphi_c(r) & r \leq R(q), \\ 0 & r > R(q). \end{array} \right. \tag{19}$$

The approach to the vacuum field when $r \to R(q)$ from below is quadratic:

$$\varphi_c(r) \cong \frac{g}{2} \left(R(q) - r \right)^2$$

for $r \lesssim R(q)$. This is typical for the V-shaped self-interactions [14].

It is clear that in the case of two point charges q_1, q_2 separated by a distance larger that $R(q_1) + R(q_2)$, the pertinent solution of the field equation

$$\Delta \varphi = g \, \mathrm{sign} \, \varphi - q_1 \delta(\vec{x} - \vec{b}) - q_2 \delta(\vec{x} + \vec{b}), \tag{20}$$

where $2|\vec{b}| > R(q_1) + R(q_2)$, is simply the sum $\varphi_{q_1}(|\vec{r} - \vec{b}|) + \varphi_{q_2}(|\vec{r} + \vec{b}|)$. The charges do not feel the presence of each other at all because there is a region of pure vacuum between them. One can surround each of the charges by a closed surface on which $T_{ik} = 0$, hence the force vanishes exactly.

When the distance between the charges is smaller than $R(q_1) + R(q_2)$, the two clouds overlap and the nonlinearity of the sign function becomes important. For simplicity, in the following we discuss the symmetric case $q_1 = q_2 \equiv q > 0$. On the basis of a hydrodynamical analogy presented below, we expect that the static solution of Eq. (20) represents a compact static cloud of the field $\varphi > 0$ which effectively screens the two charges. Let us introduce $\vec{v} = -\nabla \varphi$ and write Eq. (20) in the form

$$\nabla \vec{v} = -g + q\delta(\vec{x} - \vec{b}) + q\delta(\vec{x} + \vec{b})$$

(here $\mathrm{sign} \varphi = +1$ because $\varphi > 0$). This equation can be regarded as the governing equation for a stationary, free, irrotational flow of a liquid with unit mass density and the local velocity $\vec{v}$. The two δ terms represent the point sources of the liquid, while the constant $-g$ term represents absorption of the liquid with the rate $-g$ per unit volume. It is clear that the flux rapidly decreases with the distance from the

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sources because of the absorption, and at a certain distance, which may depend on the direction in the space, it vanishes completely. Thus, there exists a surface Σ_{2q} enclosing the sources beyond which $\vec{v} = 0$. This means the φ has a constant value $\varphi_0 \geq 0$ on Σ_{2q} and further away from the sources. Then, the function $\varphi - \varphi_0$ also obeys Eq. (20) inside Σ_{2q}, and it vanishes on Σ together with its gradient. The whole cloud has the total volume equal to $|V(2q)| = 2q/g$. This follows from Eq. (20) because

$$\int_{V(2q)} d^3x \, \Delta\varphi = \oint_{\Sigma_{2q}} d\vec{S} \, \nabla\varphi = 0, \quad \int_{V(2q)} d^3x \, \text{sign} \, \varphi = |V(2q)|.$$

Note that $|V(2q)|$ does not depend on $|\vec{b}|$.

Such a heuristic picture is quite convincing, nevertheless finding the exact shape of Σ_{2q}, and the form of φ inside the cloud is a different matter. We are able to obtain approximate formulas for the case the charges are close to each other, i.e., $|\vec{b}| \ll R(q)$. When the two charges coincide ($\vec{b} = 0$) the field has the form (19) with q replaced by $2q$, and the scalar cloud has the form of a ball. For $\vec{b} \neq 0$ we expect an axially symmetric shape. The general form of axially symmetric φ obtained from Eq. (20) under the assumption $\varphi > 0$ reads

$$\varphi(\vec{x}) = \frac{g}{6}r^2 + \frac{q}{4\pi|\vec{x} - \vec{b}|} + \frac{q}{4\pi|\vec{x} + \vec{b}|} + c_0(2q) + \delta c_0 + \sum_{l=1}^{\infty} \delta c_{2l} \, r^{2l} P_{2l}(\cos\theta), \quad (21)$$

where the polar angle θ is counted from the z-axis that is chosen to be the straight-line on which lie the two charges. It is convenient to put the origin of the coordinate system ($r = 0$) half-way in between the charges, hence $\vec{b} = (0, 0, b)$, where $b > 0$. The Legendre polynomials with odd indices do not appear in (21) because of the axial and reflection ($z \to -z$) symmetries of our set of charges. The unknown coefficients $\delta c_0, \delta c_{2l}$ in general depend on b, and they vanish for $b = 0$. The sum of the Coulomb terms is even function of b, therefore we expect that the perturbative coefficients also are even functions of b,

$$\delta c_0 = b^2 \delta c_0^{(2)} + b^4 \delta c_0^{(4)} + \ldots, \quad \delta c_{2l} = b^2 \delta c_{2l}^{(2)} + b^4 \delta c_{2l}^{(4)} + \ldots \quad (22)$$

Formula (21) holds inside the surface Σ_{2q}, which in the spherical coordinates is conveniently described by the equation

$$r^2(\theta) = R^2(2q) \, (1 + \epsilon(\theta)),$$

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where
$$\epsilon(\theta) = b^2 X^{(2)}(\theta) + b^4 X^{(4)}(\theta) + \dots$$

The coefficients (22) and the function $\epsilon(\theta)$ are to be determined from the conditions that ensure smooth matching of the solution (21) with the vacuum field $\varphi = 0$ on Σ_{2q}:
$$\varphi(r(\theta)) = 0, \quad \partial_r \varphi|_{r(\theta)} = 0.$$

These conditions are obtained from Eq. (20) in the standard way, i.e., by application of the Gauss theorem to the integral of $\triangle \varphi$ over small volumes intersecting with the surface Σ_{2q}.

Because $b \ll r(\theta)$ in the region close to Σ_{2q}, the Coulomb terms in (21) can be written as

$$\frac{q}{4\pi|\vec{x} - \vec{b}|} + \frac{q}{4\pi|\vec{x} + \vec{b}|} = \frac{q}{2\pi} \sum_{l=0}^{\infty} \frac{b^{2l}}{r^{2l+1}} P_{2l}(\cos\theta).$$

After simple calculations, we obtain from the matching conditions considered to the order b^2

$$\delta c_0^{(2)} = 0, \quad \delta c_2^{(2)} = -\frac{q}{2\pi R^5(2q)}, \quad \delta c_{2l}^{(2)} = 0 \ \text{ for } \ l > 1, \tag{23}$$

$$X^{(2)} = \frac{10}{3R^2(2q)} P_2(\cos\theta). \tag{24}$$

Calculations of the terms of the order b^4 and higher are also rather straightforward, but we will not show the results as they do not bring any really new features.

The force exerted on the charge located at $\vec{x} = -\vec{b}$ can be obtained from formula (7) with the plane $z = 0$ as the surface Σ. Because $\partial_z \varphi$ vanishes on this plane, formula (7) gives

$$F^1 = F^2 = 0, \quad F^3 = \pi \int_0^{r(\frac{\pi}{2})} dr \, r \left[(\partial_r \varphi)^2 + 2g|\varphi| \right].$$

It turns out that with the results (23), (24) one can compute F^3 up to the order b^3. The integrals are elementary, and we obtain

$$F^3 = \frac{q^2}{16\pi b^2} - \frac{gq}{3} b + \frac{2gq}{3} \frac{b^3}{R^2(2q)} + \dots.$$

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Introducing the distance between the charges $a = 2b$, and inserting $R^{-2}(2q) = (2\pi g/3q)^{2/3}$ we may also write

$$F^3 = \frac{q^2}{4\pi a^2} - \frac{gq}{6}a + \left(\frac{\pi}{36\sqrt{3}}\right)^{\frac{2}{3}} (g^5 q)^{\frac{1}{3}} \; a^3 + \dots. \tag{25}$$

The first term on the r.h.s. of this formula is the standard attractive Coulomb force, present also in formulas (11), (14) for the case of free scalar field. It dominates at small distances, i.e., when $a \ll R(2q)$. The other two terms do not have counterparts in formulas (11), (14). The self-interaction of the scalar field results in the non-analytic dependence both on the charge q and the coupling constant g.

4 The force in the one-dimensional signum-Gordon model

The one-dimensional case is interesting for two reasons. First, we will see that the total screening effect persists, in spite of the low dimension of the space. This is rather surprising because the Coulomb force in this case does not vanish at all (it is constant), and the Coulomb field of a single point charge linearly increases with the distance. Second, we can obtain exact analytic formula for the force, from which we see how it vanishes at a certain finite distance between the charges.

Let us first have a look at the field of a single point charge $q > 0$ located at $x = 0$. In one dimension the field equation has the form

$$\partial_x^2 \varphi = g \, \mathrm{sign} \, \varphi - q \, \delta(x). \tag{26}$$

Integrating both sides of this equation over infinitesimally small interval containing the point $x = 0$ we obtain the condition

$$\lim_{\epsilon \to 0+} [\partial_x \varphi(-\epsilon) - \partial_x \varphi(\epsilon)] = q. \tag{27}$$

Such discontinuity of the derivative $\partial_x \varphi$ is equivalent to the presence of point charge at $x = 0$. The pertinent solution of Eq. (26) is constructed piecewise, by first solving the homogeneous equation

$$\partial_x^2 \varphi = g \, \mathrm{sign} \, \varphi \tag{28}$$

in the two intervals $x < 0$, $x > 0$, and next imposing the condition (27) together with the continuity of φ at $x = 0$.

Boundary conditions at large $|x|$ have the form $\varphi = 0$ because then the energy density, equal to $(\partial_x\varphi)^2/2 + g\,|\varphi|$, vanishes. However there is a caveat to this because the sign φ term in the signum-Gordon equation (26) does not approach zero unless $\varphi = 0$ exactly. Therefore, the asymptotic value $\varphi = 0$ has to be reached exactly at a certain point at finite distance from the charge. It is clear that the field will stay equal to zero at larger distances only if the first derivative $\partial_x\varphi$ also vanishes at the same point.

The condition (27) suggests that $\partial_x\varphi > 0$ if $x < 0$, hence we expect $\varphi > 0$ and sign $\varphi = +1$ in a vicinity of the charge. When $x \neq 0$, Eq. (26) is simplified to

$$\partial_x^2\varphi = g, \tag{29}$$

which has the general solution of the form

$$\varphi = \frac{g}{2}x^2 + Ax + B, \tag{30}$$

where A, B are arbitrary constants. This solution should match the trivial solution $\varphi = 0$ at certain point $x = -d < 0$. The matching conditions

$$\varphi(-d) = 0, \quad \partial_x\varphi|_{x=-d} = 0$$

fix the constants A, B. In this manner we obtain the solution in the interval $-d \leq x < 0$:

$$\varphi_-(x) = \frac{g}{2}\left(x + d\right)^2. \tag{31}$$

Of course, $\varphi(x) = 0$ for all $x \leq -d$. In the region $0 < x \leq d_1$ the solution has the form

$$\varphi_+(x) = \frac{g}{2}\left(x - d_1\right)^2. \tag{32}$$

This function vanishes at $x = d_1$ together with its first derivative. In the region $x \geq d_1$ again $\varphi = 0$. The continuity of φ and the jump condition (27) for $\partial_x\varphi$ at $x = 0$ give

$$d_1 = d = \frac{q}{2g}. \tag{33}$$

The full solution, which may be called the proper field of the charge q and denoted as φ_q, has the form

$$\varphi_q(x) = \begin{cases} 0 & x \leq -d, \\ g(x+d)^2/2 & -d \leq x \leq 0, \\ g(x-d)^2/2 & 0 \leq x \leq d, \\ 0 & x \geq d. \end{cases} \tag{34}$$

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In the case two point charges q_1, q_2 located at the points $x = 0$, $x = a$, respectively, the field equation has the form

$$\partial_x^2 \varphi = g \operatorname{sign} \varphi - q_1 \delta(x) - q_2 \delta(x - a). \tag{35}$$

Now we have two conditions of the form (27)

$$\lim_{\epsilon \to 0+} [\partial_x \varphi(-\epsilon) - \partial_x \varphi(\epsilon)] = q_1, \quad \lim_{\epsilon \to 0+} [\partial_x \varphi(a - \epsilon) - \partial_x \varphi(a + \epsilon)] = q_2. \tag{36}$$

Similarly as in the case of single charge, we first solve the homogeneous equation (28) in the three intervals: $x < 0$, $0 < x < a$, $a < x$, next we impose the matching conditions with the vacuum solution $\varphi = 0$ at certain points $x = -d_- \leq 0$, $x = a + d_+$, and finally we satisfy the conditions (36) together with the conditions of continuity of φ at $x = 0$, $x = a$. The form of the solution crucially depends on the signs and relative values of the charges q_1, q_2. While it is possible to obtain exact solution for arbitrary charges, for the sake of simplicity we discuss here only the case $q_1 = q_2 = q > 0$, in which we expect a non-negative $\varphi(x)$ for all x.

Calculations analogous to the ones carried out for the single positive charge q give

$$\varphi_-(x) = \frac{g}{2}(x + d_-)^2$$

in the interval $-d_- < x < 0$, where $d_- > 0$. This solution matches the vacuum solution at $x = -d_-$, and we have $\varphi(x) = 0$ for all $x \leq -d_-$. Similarly, in the interval $a < x < a + d_+$ we have

$$\varphi_+(x) = \frac{g}{2}(x - a - d_+)^2.$$

This solution merges with the vacuum solution at $x = a + d_+$.

It remains to determine $\varphi(x)$ in the interval $0 < x < a$. We start from the general solution (30). The conditions at $x = 0, x = a$ give four equations

$$gd_-^2 = 2B, \quad gd_- - A = q, \quad gd_+^2 = ga^2 + 2Aa + 2B, \quad ga + A + gd_+ = q,$$

which determine the constants:

$$d_+ = d_- = \frac{q}{g} - \frac{a}{2}, \quad A = -\frac{ga}{2}, \quad B = \frac{g}{2}\left(\frac{q}{g} - \frac{a}{2}\right)^2.$$

Thus,

$$\varphi_0(x) = \frac{g}{2}\left[\left(x - \frac{a}{2}\right)^2 + \frac{q}{g}\left(\frac{q}{g} - a\right)\right]. \tag{37}$$

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Notice that also in this one dimensional case case the total effective charge of the scalar cloud, equal to $-g(d_- + a + d_+)$, exactly compensates the total charge $2q$ of the point sources.

It remains to check whether $\varphi_0(x) > 0$ for all $x \in (0, a)$, as assumed when passing from Eq. (28) to (29). A glance at formula (37) reveals that this is the case only when the charges are not too far from each other, namely when

$$a < a_* = \frac{q}{g}.$$

When the distance a exceed the critical value a_* the cloud of the scalar field splits into two non-overlapping clouds of the form (34). We se that $a_* = 2d$, where d is the half-width of the cloud for the single charge, formula (33).

To summarize, in the case of two identical static point charges $q > 0$ located at the points $x = 0$ and $x = a > 0$ the scalar field has the form

$$\varphi_{2q}(x) = \begin{cases} 0 & x \leq -d_-, \\ \varphi_-(x) & -d_- \leq x \leq 0, \\ \varphi_0(x) & 0 \leq x \leq a \\ \varphi_+(x) & a \leq x \leq d_+ + a, \\ 0 & x \geq d_+ + a. \end{cases} \tag{38}$$

In the one dimensional case formula (7) for the force exerted on the charge located at $x = 0$ has the form

$$F^1 = T_{11}\big|_{x=-\epsilon} - T_{11}\big|_{x=\epsilon},$$

where ϵ can be any number from the interval $(0, a)$, and $T_{11} = (\partial_x \varphi)^2/2 - g\,|\varphi|$ if $x \neq 0, a$. Using the formulas for φ_-, φ_0 we obtain $T_{11}\big|_{x=-\epsilon} = 0$, and a constant $T_{11}\big|_{x=\epsilon}$ (i.e., not dependent on ϵ). Finally,

$$F^1(a) = \frac{q^2}{2}\left(1 - \frac{a}{a_\star}\right) \tag{39}$$

for $0 < a \leq a_\star$ and $F^1(a) = 0$ if $a \geq a_\star$. Similar calculation yields the force exerted on the charge q located at $x = a$. It is equal to $-F^1(a)$, as expected. The force (39) is the attractive one. Of course, formula (39) does not apply to the overlapping charges, i.e., when $a = 0$. In this case the question about exerted forces becomes meaningless as the charges loose their identity.

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5   Summary and discussion

1. We have found that point charges interacting with the signum-Gordon classical scalar field are surrounded by a compact cloud of the field. This effect seems to be independent of the dimension of the space. We have considered the one- and three-dimensional case, but generalization to other dimensions is straightforward. The compactness is typical for the ultra-massive fields, e.g., the Q-balls and the oscillons in the signum-Gordon model are compact too. Outside the cloud the scalar field has exactly the vacuum value $\varphi = 0$, as if the charge were absent. The compactness of the cloud is directly related to the V shape of the potential $U(\varphi) = g|\varphi|$ around the vacuum field $\varphi = 0$. The ensuing $g\,\mathrm{sign}\,\varphi$ term in the field equation remains finite even if the field is arbitrarily close to the vacuum field. The only way to nullify it is to put $\varphi = 0$ exactly.

When the clouds belonging to two charges overlap, the force appears. In the three-dimensional case of two identical charges close to each other it is given by formula (25). Comparing it with formulas (11), (14) for the force in the case of free Klein-Gordon field we notice the peculiar dependence on the strength q of the charges. This is due to the non-analytic dependence on q and on the coupling constant g of the radius $R(2q) = (3q/2\pi g)^{1/3}$, as demonstrated by rewriting formula (25) in the form

$$F^3 = \frac{q^2}{4\pi a^2}\left[1 - \frac{a^3}{R^3(2q)} + \frac{1}{2}\frac{a^5}{R^5(2q)} + \mathcal{O}(a^6)\right].$$

In the one-dimensional case the force can be calculated exactly. It linearly decreases with the distance between charges until the critical distance a_* is reached, at which the two clouds are completely separated and the force vanishes.

2. In the present work we have discussed the simplest case of equal charges of same sign in order to focus on the main features of the interaction with the ultra-massive scalar field. In the one-dimensional case one can easily extend our analysis to general charges q_1, q_2. Pertinent exact formulas become much longer, but no essentially new features appear. Of course, charges of the opposite sign repel each other. In the three-dimensional case the perturbative calculations are even more cumbersome.

3. The total screening of the charges by the compact clouds effectively makes the charged particles neutral, i.e., their effective scalar charge vanishes, if measured for a distance. In a sense, the interaction mediated by our scalar field is of the contact type. The size of the scalar cloud $R(q)$ decreases with decreasing q/g,

hence it can be made very small. Such a picture can suggest ideas about dressing (some of) the known particles in such clouds of the scalar field. We hope to explore such an exotic possibility in a future work.

References

[1] See, e.g., J. E. Kim, Phys. Rep. **150**, 1 (1987); J. Khoury and A. Weltman, Phys. Rev. D **69**, 044026 (2004); C. Armendariz-Picon, T. Damour and V. F. Mukhanov, Phys. Lett. **B 458**, 209 (1999).

[2] P. Brax, C. Burrage and A. C. Davis, JCAP **1301** (2013) 020; R. Pourhasan, N. Afshordi, R. B. Mann and A.C. Davis, JCAP **1112** (2011) 005.

[3] A. C. Scott, Amer. J. Phys. **37**, 52 (1969).

[4] H. Arodź, Acta Phys. Polon. B **35**, 625 (2002).

[5] H. Arodź, P. Klimas and T. Tyranowski, Phys. Rev. E **73**, 046609 (2006).

[6] H-C. Kao, S-C. Lee and W-J. Tzeng, Physica D **107**, 30 (1997).

[7] M. J. Duncan and L. G. Jensen, Phys. Lett. B **291**, 109 (1992).

[8] H. Arodź, P. Klimas and T. Tyranowski, Phys. Rev. D **77**, 047701 (2008).

[9] H. Arodź and J. Lis, Phys. Rev. D **79**, 045002 (2009).

[10] B. Kleihaus, J. Kunz, C. Laemmerzahl and M. List, Phys. Lett. B **675**, 102 (2009).

[11] C. Adam, P. Klimas, J. Sánchez-Guillén and A. Wereszczyński, Phys. Rev. D **80**, 105013 (2009).

[12] See, e.g., D. J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Inc., Upper Saddle River, New Jersey, 1981). Section 8.2.

[13] See, e.g., J. D. Jackson, *Classical Electrodynamics (Second Edition)* (John Wiley&Sons Inc., Hoboken, New Jersey, 1975). Section 12.10.

[14] H. Arodź, Acta Phys. Polon. B **33**, 1241 (2002).